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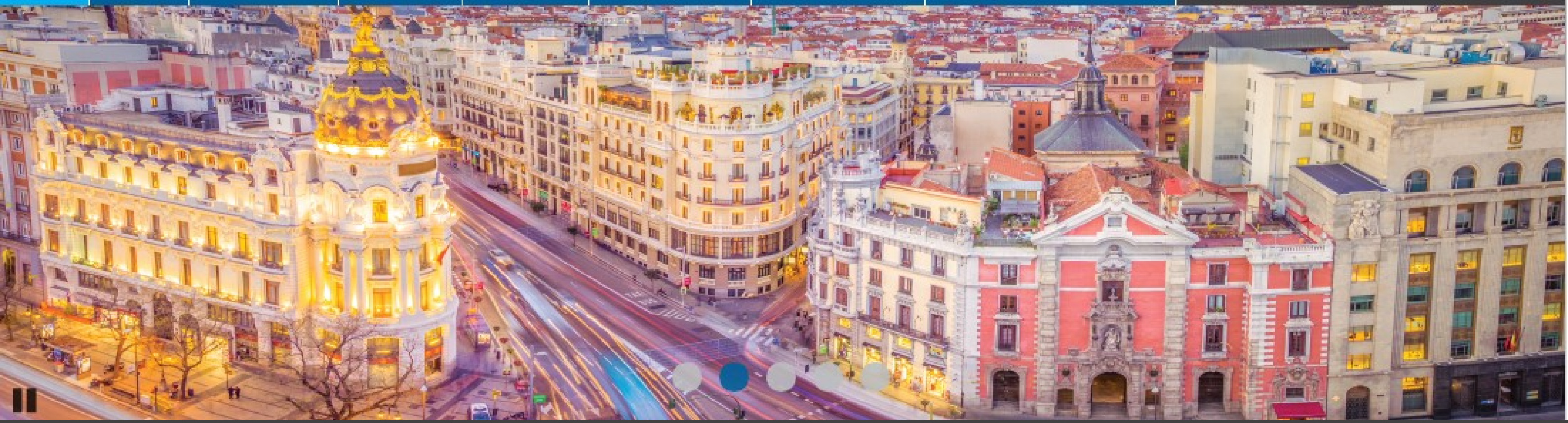

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Individual Correlation Properties and Structural Features of Periodic Complementary Sequences

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***Abstract*— Complementary sequences (CS) were considered to be used in pairs, although their property to reduce the crest factor in OFDM and MC-CDMA systems employing CS-based spreading is widely known. Their individual properties have hardly ever been studied, with one exception for the Golay sequences. In this paper, we study the individual properties of periodic CS (PCS), which are a superclass of Golay sequences. We show that PCS have remarkable correlation characteristics and unique features at their own, acting as single sequences. Although PCS are somewhat inferior to the Gold and Kasami sequences in terms of peak correlations, they are similar, and sometimes even perform better, in terms of RMS correlation values, and outnumber them by orders of magnitude. The structure of PCS enables efficient processing in applications requiring high data rates. We have also identified the unique feature of PCS which is possibility to use them to construct sets of orthogonal signals that lead to processing advantages of both complementary sequences and cyclic codes.**

Keywords*— *auto-correlation, cross-correlation, multicarrier CDMA, OFDM, PAPR, periodic complementary sequences

I. INTRODUCTION

The correlation properties of code sequences play a major part in the code design for CDMA, being responsible for the level of multiple access interference and self-interference due to multipath propagation and code acquisition. The first on is affected by the cross-correlations between different codes of the code family whereas the last two − by the periodic auto-correlations, that is the correlation between time-shifted copies of the same code sequence. A prime example of the importance of periodic correlations is in the LTE uplink reference signal sequences [1], where the cyclic shifts of the code sequence must be orthogonal or nearly orthogonal to each other. Maximum value of correlation between code sequences reflects system's worst case performance, while the root mean square (RMS) correlation values are widely used as a measure of average interference that may be produced by a particular code-set [2]. In signal processing, cross-correlation gives a measure of resemblance between the time shifted versions of signals $\alpha(t)$ and $\xi(t)$, whereas the discrete cross-correlation function (CCF) may serve as a measure of similarity between their discrete analogs.

Complementary binary sequences were originally defined as pairs of sequences with the property that the sum of their aperiodic autocorrelation functions is zero everywhere except the zero shift. Such sequences were originally introduced in [3] and are used for navigation, synchronization, in radars, and in measuring techniques. Later, aperiodic complementary sequences with more than two sequences involved were considered [4]. Periodic complementary sequences (PCS) were first introduced by Bomer and Antweiler [5]. A set of binary sequences is called a set of PCS if the sum of the periodic autocorrelation functions (PACF) of the sequences involved (often referred to as a "flock" of sequences) is zero everywhere except at zero shift. PCS include aperiodic complementary sequences as a special case.

The ultimate goal in periodic sequence design is a set of "perfect" sequences satisfying the ideal periodic correlation requirements (which means that all out-of-phase values of the autocorrelation functions are zero) [6]. In radar sensing and wireless communications, such sequences are desirable for optimal performances of a variety of applications such as spread spectrum communications, channel estimation, object detection and ranging [7]. For sequences with elements ±1, it is almost certain that only one sequence with the ideal PACF exists, i.e., (+1, +1, +1, −1). So in the binary case, PCS is a natural remedy for this situation [6]. If a flock of PCS is transmitted and afterwards correlated with twice repeated similar sequences, then the sum of the resulting periodic correlation functions is zero between the two main peaks in one period. Thus, complementary sequences are mainly used in pairs or flocks, where the complementarity of their autocorrelation functions is essential [8-13]. The individual characteristics of the sequences were never considered – with few exceptions. One of these, with promising results, is [14], where the analysis of the autocorrelation properties of Golay sequences (which are a subfamily of PCS) up to lengths N=256 was carried out by an exhaustive computer search. With that, in some applications, PCS exhibit very valuable properties, acting as single sequences. Their typical period is 2^n which is well fit to FFT-processors and other digital technique. The structure of the PCS can be defined in terms of Shapiro polynomials [15], so one more advantage is the algorithm called fast Golay correlation (FGC), which enables efficient processing for applications requiring high data rates and long code sequences [14].

It is known that complementary sequences have unique spectral properties. Their power spectrums exhibits a kind of complementarity as well: just as the autocorrelation functions of such pairs of sequences add up to a delta function, their power spectra similarly "complement" each other to a value uniformly distributed over the frequency domain. Thus, for any sequence in a pair, the spectral peaks are no more than twice the average value of the spectrum. That is, if such code sequences are applied to the IFFT block in an OFDM (or MC-CDMA) system, the Peak-to-Average Power Ratio (also known as the crest factor value), defined over discrete symbols is bounded up by 3 dB [16-20]. The great interest in complementary codes in various fields of technology, the growing variety of their applications [21, 22], as well as the indicative results of [14], are becoming an increasingly compelling reason for a detailed study of the individual properties of PCS in addition to the traditional characteristics they possess in "flocks".

In the next section, we introduce the terms and basic concepts of PCS and consider the key parameters by which their properties are evaluated in further research. In section III, we derive the basic formulas that serve as a starting point for all subsequent analysis of PCS and obtain the relations that determine the peak and RMS values of their PACF. In section IV we assess the quantitative characteristics of the PCS families. Section V examines their cross-correlations. Section VI discusses an important feature of PCS that can be used to construct orthogonal binary signal sets from them. Finally, section VII summarizes and concludes this paper.

II. Periodic Complementary Sequences: Basic Terms

Let $\{\alpha\}_N$ be a sequence of length N: $\alpha_0, \alpha_1, \ldots\alpha_{N-1}$. Then

- the sequence $\{-\alpha\}_N$ means the negation of every element of the sequence $\{\alpha\}_N$;
- $\{\xi\}_N$: $\xi_i=\alpha_i$ for i=0 mod 2, $\xi_i=-\alpha_i$ for $i\neq 0$ mod 2 denotes the alternately negated sequence $\{\alpha\}_N$;
- $\{\xi\}_N\equiv\{\alpha_s\}_N$: $\xi_i=\alpha_{(i+s)\bmod N}$ denotes the sequence $\{\alpha\}_N$ shifted cyclically by s elements;
- $\{\xi\}_N$ where $\xi_i=\alpha_{N-i}$, except $\xi_0=\alpha_0$, denotes the reverse of the sequence $\{\alpha\}_N$;
- $\{\xi\}_N$: $\xi_i=\alpha_{N-1-i}$, denotes the reflected sequence $\{\alpha\}_N$.

The common way to create a sequence complementary with respect to a given sequence $\{\alpha\}_N$ is the alternate negating: $\beta_i=(-1)^i\cdot\alpha_i$, i=0,1,...N–1. The negation, reflection (reverse) and any shift of either or both of these sequences will not change their complementarity since all these do not change the PACF of the sequences. That is, for $\alpha_0=\pm1$, $\beta_0=-\alpha_0$, and s=0,1,...N/2–1, at the n-th iteration we have

$$\{\alpha\}_N : \alpha_0,\beta_0,\alpha_1,\beta_1,\ldots\alpha_{N/2-1},\beta_{N/2-1} \equiv \{\alpha\}_{N/2}\#\{\pm\beta_s\}_{N/2}^{(*)}, \quad (1)$$

where $\{\alpha\}_{N/2}$ and $\{\beta\}_{N/2}$ are PCS of length N/2, "#" denotes their interleaving, "–" denotes negating of each element of a sequence, and (*) means its possible reverse.

Periodic CCF (PCCF) is a measure of similarity of two different sequences as a function of the displacement m of one relative to the other, whereas the periodic autocorrelation function (PACF) is a measure of the similarity between two shifted copies of the same sequence:

$$\Xi_N(m)=\sum_{i=0}^{N-1}\alpha_i\xi_{i+m}\ ;\ R_N(m)=\sum_{i=0}^{N-1}\alpha_i\alpha_{i+m}\ ,\ \alpha_i\,,\xi_j\in\{-1;1\}.$$

Let $\{\gamma\}_{2N}$ be an interleaved sequence of $2N$ elements which is obtained via merging (by alternately mixing) two sequences $\{\alpha\}_N$ and $\{\beta\}_N$:

$$\{\gamma_0,\gamma_1,\gamma_2,\ldots\gamma_{2N-1}\}\equiv\{\alpha_0,\beta_0,\alpha_1,\beta_1,\ldots\alpha_{N-1},\beta_{N-1}\} : \{\alpha\}_N\#\{\beta\}_N .$$

Let us denote the sequence $\{\beta\}_N$ complementary to $\{\alpha\}_N$ as $\{\beta\}_N=\mathbf{C}\{\alpha\}_N$. By definition, its main property is

$$R_{\beta,N}(m)=-R_{\alpha,N}(m) \text{ for any } m\neq 0.$$

If we choose a sequence $\{\alpha\}_N$ in such a way that

$$R_{\alpha,N}(2m)=0\ \forall\ m=1,\ldots,N/2-1, \quad (2)$$

then a sequence $\{\beta\}_N$ such that $\beta_i=(-1)^i\cdot\alpha_i$, i=0, ..., N–1, is complementary to $\{\alpha\}_N$, since for m=0,...,N/2–1:

$$R_{\beta,N}(2m)=R_{\alpha,N}(2m) \text{ and } R_{\beta,N}(2m+1)=-R_{\alpha,N}(2m+1).$$

In what follows, we say that a sequence of N elements with property (2) belongs to G_N (the family of conventional or "regular" PCS). As (2) is always true for any binary sequence of 4 elements the sum of which is even (e.g., {–1, 1, 1, 1}), there are $2^4/2=8$ such sequences. Any regular sequence of length $N=2^n$ belonging to G_N family can be constructed by interleaving a pair of PCS of length N/2 that belong to $G_{N/2}$ and are complementary with respect to each other. The sequence $\{\gamma\}_8\equiv\{\alpha\}_4\#\{\beta_s\}_4$ built in accordance with (1) has the property (2) by virtue of

$$R_{\gamma,N}(2m)=R_{\alpha,N/2}(m)+R_{\beta,N/2}(m). \quad (3)$$

Periodic complementary sequence of length $N=2^k$ can always be obtained from an initial PCS of length N=4 by k–2 fold alternate negation and interleaving. Indeed, if the PACF of a sequence $\{\alpha\}_N$ satisfies (2), then its alternate negation is the sequence $\{\beta\}_N$, the PACF $R_{\beta,N}(m)=-R_{\alpha,N}(m)$ of which takes the form (2). By interleaving $\{\alpha\}_N$ and $\{\beta\}_N$, we obtain the sequence $\{\gamma\}_{2N}\equiv\{\alpha\}_N\#\{\beta\}_N$ with property (2) by virtue of (3). It is also obvious that the sequence complementary to the sequence $\{\gamma\}_{2N}$ always exists. Thus, if (2) holds for some sequence, then the latter belongs to G_N, which is generated by sequences from the family $G_{N/2}$.

Let $\Xi_N(m)$ denote a periodic cross-correlation between any two sequences of length N in the family G_N, whereas $\Theta_N(m)$ is the cross-correlation function "within" a pair of complementary sequences (i.e., between two sequences that belong to G_N and are complementary to each other). For the sake of discussion below, we also use the following terms and definitions. So, for a sequence of length N,

- R_N and Θ_N are the maximum PACF value and the maximum PCCF value (for any mutual shifts "inside" a pair of PCS), respectively;
- $\widehat{R}_N$ ($\widehat{\Theta}_N$) and $\breve{R}_N$ ($\breve{\Theta}_N$) are the maximum PACF (PCCF) values for even and all odd m, respectively.

Note that the mean value of PACF sidelobes $\langle R_N(m)\rangle=0$ for $m\neq0$ (since for any $\{\alpha\}_N\in G_N$ there always exists a sequence $\{\beta\}_N=\mathbf{C}\{\alpha\}_N$), and, in exactly the same way, the mean value of PCCF $\langle\Theta_N(m)\rangle=0$ (since for any $\{\alpha\}_N\in G_N$ there is always $\{\beta\}_N\equiv\{-\alpha\}_N$). Therefore, the mean squares of the sidelobes of both correlation functions coincide with their variances (σ^2_N and D_N^2, respectively).

III. Autocorrelation properties of PCS

It is easy to prove that for any sequence of length N constructed from a pair of complementary PCS of length N/2 in accordance with (1), the following relations hold:

$$\Theta_N(2m) = R_{\alpha,N/2}(m) - R_{\beta,N/2}(m) = 2R_{N/2}(m),\ m\neq0; \quad (4)$$

$$R_N(2m+1)=\Theta_{N/2}(N/2-1-m)+\Theta_{N/2}(m)\ ; \quad (5)$$

$$\Theta_N(2m+1)=\Theta_{N/2}(N/2-1-m)-\Theta_{N/2}(m). \quad (6)$$

Hence, in particular, it follows that

$$\sum_{m=0}^{N/2-1}\Theta_N(2m+1)=\sum_{m=0}^{N/2-1}\left[\Theta_{N/2}(m)-\Theta_{N/2}(m)\right]=0\ . \quad (7)$$

In what follows, we will also use two other formulae:

$$\Theta_N^{(s)}(m)=\Theta_N(m+s)\,, \quad (8)$$

where $\Theta_N^{(s)}(m)$ is a PCCF of sequences $\{\alpha\}_N$ and $\{\beta_s\}_N$ (this equality is obvious); and

$$\sum_{m=0}^{N-1}\Xi_N^2(m)=\sum_{m=0}^{N-1}R_{\alpha,N}(m)\cdot R_{\beta,N}(m), \qquad (9)$$

which is also not hard to prove using the convolution theorem. Proved for any two arbitrary sequences, this equality is obviously also true for a more special case of sequences complementary to each other.

The cross-correlation matrix of $\{\alpha\}_N$ and $\{\beta_s\}_N$, where s=0,..., N–1 (the matrix that contains the cross-correlations between shifted copies of these sequences as its elements) is a left circulant (or reverse circulant) Toeplitz matrix:

$$\Theta(s,m)=\begin{bmatrix}\Theta_0 & \Theta_1 & \Theta_2 & \dots & \Theta_{N-1}\\ \Theta_1 & \Theta_2 & \Theta_3 & \dots & \Theta_0\\ \dots & \dots & \dots & \dots & \dots\\ \Theta_{N-1} & \Theta_0 & \Theta_1 & \dots & \Theta_{N-2}\end{bmatrix}.$$

Let a sequence $\{\gamma\}_N\equiv\{\alpha\}_{N/2}\#\{\beta_s\}_{N/2}$ be an interleaving of a complementary pair of sequences of length $N/2$. Being an extended form of the PACF, its autocorrelation matrix $R(s,m)$ is also circulant. According to (5), for odd m its columns are linear combinations of symmetrically opposite columns of the matrix $\Theta(s,m)$ like the $\Theta_{N/2}(m)+\Theta_{N/2}(n)$, $m+n=1 \pmod 2$. Its even columns are zero, so we find the sum of the $R_N^2(s,m)$ values for odd m, averaging it over all s and m, s=0,...,N/2–1, m=1,..., N–1:

$$\sigma_N^2=\frac{2}{N(N-1)}\sum_{s,m=0}^{N/2-1}\sum R_N^2(s,2m+1).$$

Taking into account (5) and rearranging the terms in the following expression we obtain:

$$\sigma_N^2=4\frac{2}{N(N-1)}\sum_{m,n=0}^{N/4-1}\sum\left[\Theta_{N/2}(2m)+\Theta_{N/2}(2n+1)\right]^2.$$

As each pair of (m,n) occurs four times there and, having in mind the property (7), thence:

$$\sigma_N^2=\frac{8}{N(N-1)}\left[\frac{N}{4}\cdot\sum_{m=0}^{N/4-1}\Theta_{N/2}^2(2m)+\right.$$

$$\left.2\sum_{m=0}^{N/4-1}\Theta_{N/2}(2m)\sum_{n=0}^{N/4-1}\Theta_{N/2}(2n+1)+\frac{N}{4}\sum_{n=0}^{N/4-1}\Theta_{N/2}^2(2n+1)\right]=$$

$=\frac{2}{N-1}\cdot\sum_{m=0}^{N/2-1}\Theta_{N/2}^2(m)$. Then it follows from (9) that

$$\sum_{m=0}^{N/2-1}\Theta_{N/2}^2(m)=\sum_{m=0}^{N/2-1}R_{N/2,\alpha}(m)\cdot R_{N/2,\beta}(m)=$$

$$=\left(\frac{N}{2}\right)^2-\sum_{m=1}^{N/2-1}R_{N/2}^2(m)=\frac{2}{N-1}\left[\left(\frac{N}{2}\right)^2-\left(\frac{N}{2}-1\right)\cdot\sigma_{N/2}^2\right],$$

wherefrom

$$\sigma_N^2=\frac{N^2}{2(N-1)}-\frac{N-2}{N-1}\cdot\sigma_{N/2}^2. \qquad (10)$$

From (10), knowing the initial term $\sigma^2_4=0$, one can obtain all subsequent terms for any $N=2^k$, where k is an integer. We also can find the relationship between the variance of PACF of a periodic complementary sequence and the variance of PCCF between a pair of PCS:

$$D_N^2=\frac{1}{N}\cdot\sum_{m=0}^{N-1}\Theta_N^2(m)=N-\frac{N-1}{N}\cdot\sigma_N^2. \qquad (11)$$

Finally, from (10) and (11) we derive the relationship

$$D_N^2=N-D_{N/2}^2. \qquad (12)$$

As it was noted above, when looking over the values of $R_N(s,2m+1)$ over all s and m, all possible pairs of elements of the matrix $\Theta(s,m)$ are summed, where $\Theta_{N/2}(m)+\Theta_{N/2}(n)$, $m+n=1 \pmod 2$. Therefore, for any sequence $\{\gamma\}_N$, there always exists a maximum value of the PACF sidelobe: $\breve{R}_N=\widehat{\Theta}_{N/2}+\widehat{\Theta}_{N/2}$ (since $\widehat{R}_N=0$ by virtue of (2)).

For any $\{\beta\}_{N/2}\in G_{N/2}$ the sequence $\{-\beta\}_{N/2}$ also belongs to the family $G_{N/2}$, whereas the negation of either of the two sequences, $\{\alpha\}_{N/2}$ and $\{\beta_s\}_{N/2}$, switches their PCCF to the opposite sign. Thus, according to (6), the maximum value of cross-correlation between sequences $\{\gamma\}_N$ and $\{\mu\}_N$ of length N, where $\{\mu\}_N=\mathbf{C}\{\gamma\}_N$, always exists and is equal to $\breve{\Theta}_N=\widehat{\Theta}_{N/2}+\breve{\Theta}_{N/2}$. Having in mind the above and taking into account formula (4) let us write:

$$\widehat{\Theta}_N=2\breve{R}_{N/2}\ \text{ and, }\ R_N=\breve{\Theta}_N=\widehat{\Theta}_{N/2}+\breve{\Theta}_{N/2}.$$

From the last two equalities it follows that

$$R_N=R_{N/2}+2R_{N/4}. \qquad (13)$$

We transform (13) by setting $N=2^k$, $R_N=r_k$. Then it takes the form $r_k=r_{k-1}+2r_{k-2}=(r_{k-2}+2r_{k-3})+2r_{k-2}=$

$=4r_{k-2}+(-1)^{k-j}(r_{j+1}-2r_j)$. Let in turn,

$j=2$ ($r_2=R_4=0$, $r_3=R_8=4$), then $r_k=4\cdot\left(r_{k-2}+(-1)^k\right)$.

Theorem 1. The PACF sidelobe variance of any PCS is linearly related to its maximum absolute sidelobe value:

$$\sigma_N^2=\frac{N}{N-1}R_N. \qquad (14)$$

Lemma 1. If (14) holds for sequences of the families $G_{N/4}$ and $G_{N/2}$, then it is also true for sequences of G_N family.

Proof. Let's substitute (14) into (13) and assume that the expression below is true (by the condition of Lemma 1):

$$R_N=\frac{N/2-1}{N/2}\cdot\sigma_{N/2}^2+2\frac{N/4-1}{N/4}\cdot\sigma_{N/4}^2.$$

The second term is easily reduced to the form:

$$2\frac{N/4-1}{N/4}\sigma_{N/4}^2=\frac{N}{2}-2\frac{N-2}{N}\left[\frac{(N/2)^2}{2(N/2-1)}-\frac{N/2-2}{N/2-1}\sigma_{N/4}^2\right]=$$

$\frac{N}{2}-2\frac{N-2}{N}\sigma_{N/2}^2$, whence, considering (10) we derive the

closed-form expression $R_N=\frac{N}{2}-\frac{N-2}{N}\sigma_{N/2}^2=$

$$= \frac{N-1}{N}\left[\frac{N^2}{2(N-1)} - \frac{N-2}{N-1}\sigma^2_{N/2}\right] = \frac{N-1}{N}\sigma^2_N .$$

This proves Lemma 1.

It is easy to verify that (14) holds for N=4 (Θ_4=2, R_4=0, $\sigma_4^2 = 0$) and, for N=8 (Θ_8=8, R_8=4, $\sigma_8^2 = 32/7$). Therefore, according to Lemma 1 (14) is true for all $N=2^k$, where k is an integer. This completes the proof of the theorem. Hence, (11) can be rewritten: $D_N^2 = N - R_N$.

Comparing this equality with (12), we get $R_N = D^2_{N/2}$. This means that the peak PACF sidelobe value of sequence $\{\gamma\}_{2N} \equiv \{\alpha\}_N \# \{\beta\}_N$, $\{\beta\}_N \equiv \mathbf{C}\{\alpha\}_N$, is equal to the variance of the PCCF of sequences $\{\alpha\}_N$ and $\{\beta\}_N$. What is also directly follows from these two equations is the relation between the length of PCS and the maximum absolute level of their PACF sidelobe: $R_N+R_{2N}=N$.

The last relationship allows us replacing (13) with a non-recursive formula for the maximum values of the PACF:

$$R_N = \frac{N}{4} + R_{N/4}\text{, consequently, } r_k = 2^{k-2} + r_{k-2} =$$

$$= 2^{k-2} + 2^{k-4} + r_{k-4} = \ldots = 2^{k-2} + 2^{k-4} + \ldots + 2^2 + r_2 =$$

$$0 + 2^2 + 2^4 + \ldots + 2^{k-2} = \sum_{i=1}^{k/2-1} 4 \cdot 4^{i-1} = \frac{2^k - 4}{3} \text{ for even } k;$$

$$r_k = 2^{k-2} + 2^{k-4} + \ldots + 2^3 + r_3 =$$

$$4 + 2^3 + 2^5 + \ldots + 2^{k-2} = \frac{2^k + 4}{3} \text{ for odd } k.$$

Hence we get $r_k = \frac{1}{3}\left(2^k - 4(-1)^k\right)$ or, which is the same,

$$R_N = \frac{1}{3}\left(N - 4(-1)^k\right),\ N=2^k. \qquad (15)$$

Although the maximum PACF sidelobe values for PCS are greater than those of the Gold or Kasami sequences, large sidelobes are rare (this is indicated by the smallness of σ^2_N). For large N, a simplified assessment can be applied:

$$\sigma_N^2 \approx (N - 4(-1)^k)/3 . \qquad (16)$$

For Gold sequences of similar length, the average square of the PACF sidelobe is nearly thrice this value: N+1−1/N [23].

IV. Quantitative Characteristics and Bounds of the Number of Sequences in the PCS Family

Even-numbered shifts of any PCS are orthogonal to each other according to (2). Basically, there are two ways to build a periodic complementary sequence. In the first, PCS of length N (i.e., sequences that belong to G_N) are constructed according to procedure (1) from sequences that belong to $G_{N/2}$. In what follows we will refer to such sequences as "regular" PCS. However, some of the PCS (let's call them "irregular" PCS) may be obtained outside the scope of (1). One can get them by alternating complementary sequences $\{\alpha\}_{N/2} \notin G_{N/2}$, $\{\beta\}_{N/2} \notin G_{N/2}$ (i.e., $R_\beta(m)=-R_\alpha(m)$, $m\neq 0$, but, $R_\alpha(2m)\neq 0$, $R_\beta(2m)\neq 0$ for some of m), starting from N=16. Irregular PCS are also a subset of the family G_N: $S_N \in G_N$.

Let Q_N be a total number of sequences in the family G_N – both regular and irregular: $Q_N = Q_N^{(R)} + Q_N^{(S)}$. There are 64 PCSs of length 8, generated from the PCS of length N=4 (e.g., $\{\alpha\}_4$: −1, 1, 1, 1). Among the sequences of length 8, only regular PCS exist: $Q_8 = Q_8^{(R)} = 64$. From these latter, in turn, $Q_{16}^{(R)} = 2^{10}$ regular PCSs of length 16 can be built with all the negations, permutations and reciprocal shifts of subsequences with even and odd elements, respectively: $\{\alpha\}_{N/2}$ and $\{\beta\}_{N/2}$. It is obviously, that $Q_N^{(R)} = 2N \cdot Q_{N/2}$.

Besides, complementary pairs of sequences of length 8 can be found outside G_8: for instance, + + + + − + − + and, − + + − + + − −. The total number of different pairs $(\{\alpha\}_8, \{\beta\}_8)$ (with all possible cyclic shifts and negating of each of the sequences of the pair) is 2^9. Therefore, there are $Q_{16}=2^{10}+2^9=1536$ distinct sequences in the family G_{16}.

Most PCS of length 32 are constructed from sequences belonging to G_{16}. This results in $Q_{32}^{(R)} = 2 \cdot 2N \cdot Q_{16} = 3 \cdot 2^{16}$ PCSs in total. At the same time, for N=16 there is also another set of periodic complementary sequences such that $S_{16} \notin G_{16}$, which is specified in a compact form by a set of 6 generating sequences (see Table 1, where "0" represents +1 and "1" stands for −1). The right column of the table shows the number of different sequences with the same PACF.

TABLE I
Generating Sequences for Constructing Irregular Comlementary Sequences of Length N=32

generating sequence	PACF $R(m)$, m=0,1,…, N–1	number of sequences
1101010110000000	16, 0, 4, 0, 0, 0, –4, 0, 0, 0, –4, 0, 0, 0, 4, 0	* 32
1110101001000000	16, 0, 4, 0, 0, 0, –4, 0, 0, 0, –4, 0, 0, 0, 4, 0	64
1011101000010000	16, 0, 4, 0, 0, 0, –4, 0, 0, 0, –4, 0, 0, 0, 4, 0	* 32
1100101001100000	16, 0, –4, 0, 0, 0, 4, 0, 0, 0, 4, 0, 0, 0, –4, 0	* 32
1100010110010000	16, 0, –4, 0, 0, 0, 4, 0, 0, 0, 4, 0, 0, 0, –4, 0	64
1110001001001000	16, 0, –4, 0, 0, 0, 4, 0, 0, 0, 4, 0, 0, 0, –4, 0	* 32

Some sequences are self-conjugate (those denoted with an asterisk in Table 1) which means that the reverse of a sequence coincides with one of its shifts, i.e., does not result in a new sequence. Such sequences owe their name to the peculiarities of their spectra, such that the complex conjugation of the spectrum of the sequence is identical to the spectrum of its time shift (it differs from the original one only by multiplication by a complex exponent). As follows from Table 1, there are 256 sequences in S_{16} in total, so the number of PCS of length 32 they produce is

$$Q_{32}^{(S)} = 2 \cdot (32 + 64 + 32)^2 = 4N \cdot 256 = 2^{15}\text{. Hence,}$$

$$Q_{32} = Q_{32}^{(R)} + Q_{32}^{(S)} = (2^{17} + 2^{16}) + 2^{15} = 7 \cdot 2^{15} = 229376 .$$

This is a scalable synthesis: for arbitrary N, a lower bound $Q_N \geq 2NQ_{N/2}$ can be used. The number of different PCS of particular length along with peak PACF values are shown in Table 2. The number of PCS-based complementary cyclic codes is obviously equal to the number of PCS, excluding all cyclic shifts and negations of the latter: $K_N = Q_N / 2N$.

TABLE II
PARAMETERS OF PACF AND THE NUMBER OF PCS

N	R_N	σ^2_N	Q_N
4	0	0	2^3
8	4	4.571	2^6
16	4	4.266	3×2^9
32	12	12.387	$7\cdot2^{15}$
64	20	20.317	$\geq7\cdot2^{22}$
128	44	44.346	$\geq7\cdot2^{30}$
256	84	84.329	$\geq7\cdot2^{39}$

V. CROSS-CORRELATION PROPERTIES OF PCS

In examining the cross-correlation properties of PCS, we rely on their key feature – "complementarity": the property of being an "element" of a complementary pair. Therefore, the results of such a study are applicable to a whole variety of PCS, including regular and irregular ones. Let us denote the total number of complementary cyclic codes in such a set by the symbol K_N^+. This set can be conditionally halved into two subsets in such a way that any sequence from one subset (we denote it by L^+) will have a complementary one in the other subset M^+:

$\{\lambda\}_N\in L^+ \Leftrightarrow \exists\ \{\mu\}_N : \{\mu\}_N\in M^+,\ \{\mu\}_N=\mathbf{C}\{\lambda\}_N$.

The PACFs of the sequences $\{\lambda\}_N$, $\{\mu\}_N$ have opposite signs for $m\neq0$. Let us find the average sum of the squares of the PCCF values in the PCS family with respect to arbitrary PCS $\{\alpha\}_N$. Since the number of cyclic codes in the subsets is $Q(L^+)=Q(M^+)=K_N^+/2$, then in view of (9) we have

$$\left\langle\sum_{m=0}^{N-1}\Xi^2_{\alpha\xi}(m)\right\rangle=\frac{1}{K_N^+}\left(\sum_{\lambda\in L^+}\sum_{m=0}^{N-1}\Xi^2_{\alpha\lambda}(m)+\sum_{\mu\in M^+}\sum_{m=0}^{N-1}\Xi^2_{\alpha\mu}(m)\right)=$$

$$=\frac{1}{K_N^+}\left(\sum_{\gamma\in L^+}\sum_{m=0}^{N-1}R_\alpha(m)\cdot R_\lambda(m)+\sum_{\mu\in M^+}\sum_{m=0}^{N-1}R_\alpha(m)\cdot R_\mu(m)\right)=$$

$$=\frac{1}{K_N^+}\left[\sum_{\lambda\in L^+}\left(\sum_{m=1}^{N-1}R_\alpha(m)\big(R_\lambda(m)-R_\lambda(m)\big)\right)+K_N^+\cdot N^2\right]=N^2.$$

Since one of the sequences designated as $\{\xi\}_N$ is actually the sequence $\{\alpha\}_N$ itself, we must account for the correction value which converts one of the PCCFs above to PACF. As a result we get

$$\sigma^2_\Xi=\frac{1}{N}\times\frac{1}{K_N^+-1}\left(K_N^+\times\left\langle\sum_{m=0}^{N-1}\Xi^2_{\alpha\xi}(m)\right\rangle-\sum_{m=0}^{N-1}R^2_\alpha(m)\right)=$$

$$=\frac{1}{N(K_N^+-1)}\left(N^2K_N^+-N^2-(N-1)\cdot\sigma^2_N\right)=$$

$$=\frac{(K_N^+-1)\cdot N^2-(N-1)\cdot\sigma^2_N}{N(K_N^+-1)}=N-\frac{N-1}{N}\cdot\frac{\sigma^2_N}{K_N^+-1}.$$

As noted above, this equality was obtained without loss of generality and, therefore, is valid for all PCS. According to (14) and (15), the PACF sidelobe variance of the PCS of length N is $\sigma^2_N=N\cdot\dfrac{N-4\cdot(-1)^n}{3\cdot(N-1)}$, whence we finally have

$$\sigma^2_\Xi=N-\frac{N-4\cdot(-1)^n}{3\cdot(K_N-1)}. \qquad (17)$$

The RMS values of the PCCF along with the number of possible codes of particular length are tabulated in Table 3.

TABLE III
GOLD AND KASAMI SEQUENCES VS PCS

N_{Gold} / N_{PCS}	K_N (number of codes)			**rms CCF** / ACF	
	Gold	Kasami	PCS	Gold	PCS
7 / 8	9	—	4	**2.803**	**2.582** 2.138
15 / 16	17	67	48	**3.992**	**3.989** 2.065
31 / 32	33	—	3584	**5.654**	**5.656** 3.519
63 / 64	65	519	229376	**7.999**	**7.999** 4.507
127 / 128	129	—	29360128	**11.313**	**11.313** 6.659
255 / 256	257	4111	7516192768	**15.999**	**15.999** 9.183

For $N\leq32$, (17) is confirmed by an exhaustive computer search along with the calculation of cross-correlations of the PCS. With the aim of comparison, the relevant parameters of the most traditional PN-code families (Gold codes and Large sets of Kasami sequences) are given here. Whereas the latter outperform PCS in terms of peak correlation value (for $N=2^n-1$, the highest absolute cross-correlation in these sets of codes is $2^{(n+2)/2}+1$ for even n and $2^{(n+1)/2}+1$ for odd n), the PCS are many times superior to them numerically (there are 2^n+1 different Gold sequences and $2^{n/2}\times(2^n+1)-1$ is the size of the Large set of Kasami sequences).

Moreover, large PCS correlations are very rare which is indicated by the smallness of their average squares: see (16) and (17). With that, it is the RMS value of ACF and CCF that is the parameter that dominates CDMA performance at average signal-to-noise power ratios [24].

VI. COMPLEMENTARY "TWINS"

Another important feature of PCS is their suitability for constructing complete sets of orthogonal signals. Double cyclic shifts of any sequence from the family G_N give us a subset of orthogonal waveforms of size $N/2$. Let's determine the condition under which two such subsets of waveforms are orthogonal to each other. Let there be sequences $\{\alpha\}_N\in G_N$, $\{\beta\}_N\in G_N$. Then the polynomial representation of their PCCF has the form

$$\Theta_N(Z)=\sum_{m=0}^{N-1}\Theta(m)Z^{mk}=A^*(Z^k)\cdot B(Z^k), \qquad (18)$$

where $A(Z)$ and $B(Z)$ are polynomial representations of the sequences $\{\alpha\}_N$ and $\{\beta\}_N$, respectively, $Z^k=-j2\pi k/N$. With a minor rearrangement of even and odd elements on the right side of (18), these polynomials can be reduced to the forms

$$A(Z)=A_0(Z^2)+Z\cdot A_1(Z^2);\quad B(Z)=B_0(Z^2)+Z\cdot B_1(Z^2),$$

where, for example $A_0(Z^2)=\sum_{i=0}^{N/2-1}\alpha_{2i}Z^{2i}$.

For even samples according to (18), one can get

$$\sum_{m=0}^{N/2-1}\Theta_N(2m)\cdot Z^{2m}=A_0^*(Z^2)\cdot B_0(Z^2)+A_1^*(Z^2)\cdot B_1(Z^2),$$

which is actually a decimated (by a factor of 2) Discrete Fourier Transform of the PCCF. Setting the latter to zero for even shifts (or, setting to zero Fourier coefficients on the left side of the equation, which means the same) (i.e., $\Theta(2m)=0$, $m=0,1,...N/2-1$) we can find out the structures of the polynomials $A(Z)$ and $B(Z)$ that ensure the orthogonality of mutual even shifts of the sequences $\{\alpha\}_N$ and $\{\beta\}_N$:

$$\frac{B_1(Z^2)}{B_0(Z^2)} = -\left(\frac{Z^s \cdot A_0(Z^2)}{Z^s \cdot A_1(Z^2)}\right)^*, \text{ where } s \text{ is an integer.}$$

Here, the multiplication of the polynomials $A_0(Z^2)$, $A_1(Z^2)$ by the factor Z^s corresponds to cyclic shifts of even and odd subsequences by s elements (to cyclic shift of the sequence $\{\alpha\}_N$ by $2s$ elements). Positive s means left shift direction. Hence it follows that the polynomials $A(Z)$, $B(Z)$ of sequences $\{\alpha\}_N$ and $\{\beta\}_N$, orthogonal to each other, are related by formula $B(Z) = \pm Z^{2S} \cdot \left(A_0(Z^2) - Z \cdot A_1(Z^2)\right)^*$.

The Inverse DFT from the right-hand side of $B(Z)$ is the sequence formed from $\{\alpha\}_N$ by alternate negating. Complex conjugation (*) means the reverse of the order of elements of the sequence. Since neither the reverse nor the reflection of $\{\alpha\}_N$ changes its PACF, then $\{\beta\}_N \in G_N$. The even shifts of α- and β-sequences (we will call them complementary twins) produce two mutually orthogonal signal subsets of $N/2$ orthogonal waveforms each, which gives us a complete set of orthogonal waveforms of size N. Below is an example of such a set of orthogonal waveforms of length N=8:

$$\text{"}\alpha\text{-code"}: \begin{bmatrix} - & - & + & - & + & + & + & - \\ + & - & + & + & + & - & - & - \\ + & + & + & - & - & - & + & - \\ + & - & - & - & + & - & + & + \end{bmatrix}$$

$$\text{"}\beta\text{-code"}: \begin{bmatrix} - & + & + & + & + & - & + & + \\ + & + & + & - & + & + & - & + \\ + & - & + & + & - & + & + & + \\ + & + & - & + & + & + & + & - \end{bmatrix}$$

These signal sets, in particular, are an extension of the known Welti codes (considering the isomorphism between the sequences of Golay and the codes of Welti [25]).

VII. Conclusions

PCSs represent special group of codes with large family size and good correlation properties. Their scalability enables design a bandwidth and data rate adaptive CDMA system with low crest factor, since the length and size of PCS sets can be adjusted according to the number of active users. Their structure leads to both fast processing algorithms (such as FFT and FGC) and a method for generating new sets of orthogonal codes. As a topic for further research, the fact that PCS can be seen as good individual sequences could open up a broader class of new applications for them. Another open issue is the expression for the maximum cross-correlation value of the PCS, which has yet to be derived.

References

[1] E. B. Dahlman, S. Parkvall and J. Sköld, "Uplink Physical-Layer Processing," in *4G LTE Advanced Pro and The Road to 5G*, 3rd ed., Elsevier Academic Press, 2016, pp. 169–225.

[2] Sarwate D.V., "Mean-square correlation of shift register sequences", IEE Proc. vol. 131, Part F, no. 2, pp. 101–106, April 1984.

[3] M.J.E. Golay, "Complementary series," IRE Trans., vol. IT-7, no. 2, pp. 82–87, April 1961, DOI: 10.1109/tit.1961.1057620.

[4] C. Tseng and C. Liu, "Complementary sets of sequences," IEEE Trans. Inform. Theory, vol. IT-18, pp. 644–665, Sep. 1972.

[5] L. Bomer and M. Antweiler, "Periodic complementary binary sequences," IEEE Trans. on Inform. Theory, vol. IT-36, no. 6, pp. 1487–1494, Nov. 1990, DOI: 10.1109/18.59954.

[6] K. Feng, P. J.-S. Shiue and Q. Xiang, "On aperiodic and periodic complementary binary sequences," ieee trans. on inform. theory, vol. it-45, no. 1, pp. 296–303, jan. 1999, DOI: 10.1109/18.746823.

[7] P. Fan and M. Darnell, *Sequence Design for Communications Applications*. New York, NY, USA: Wiley, 1996.

[8] S.M. Tseng and M.R. Bell, "Asynchronous multicarrier DS-CDMA using mutually orthogonal complementary sets of sequences," IEEE Trans. on Commun., vol. 48, pp. 53–59, Jan. 2000.

[9] H.-H. Chen, J.-F. Yeh, and N. Suehiro, "A Multicarrier CDMA architecture based on orthogonal complementary codes for new generations of wideband wireless communications," IEEE Commun. Mag., vol. 39, no 10, pp. 126–135, Oct. 2001.

[10] H.-H. Chen, D. Hank, M. E. Magana, and M. Guizani, "Design of next-generation CDMA using orthogonal complementary codes and offset stacked spreading," IEEE Wireless Commun., vol. 14, no. 3, pp. 61–69, Jun. 2007, DOI: 10.1109/MWC.2007.386614.

[11] S. Han, M. Liu, W. Meng and Y. Zhang, "Transmission scheme and performance analysis of complementary coded full duplex communication system," Proc. 2016 IEEE Global Communications Conference, pp. 168–172, 4-8 Dec. 2016, Washington, DC, USA.

[12] G. Han and J. Choi, "Radar Imaging Based on IEEE 802.11ad Waveform," Proc. 2020 IEEE Global Communications Conference, 7-11 Dec. 2020, Taipei, Taiwan.

[13] Z. Liu, Y.L. Guan, and H.-H. Chen, "Complementary coded scrambling multiple access and its performance in downlink MIMO channels," IEEE Trans. Wireless Commun., vol. 17, no. 2, pp. 835–847, Feb. 2018, DOI: 10.1109/TWC.2017.2771780.

[14] S. Budisin, "Golay complementary sequences are superior to PN sequence," IEEE International Conf. on Systems Engineering, 1992, Sep. 1992, pp. 101-104, DOI: 10.1109/ICSYSE.1992.236932.

[15] H. S. Shapiro. "Extremal problems for polynomials and power series", M.S. thesis, Massachusetts Institute of Technology, 1951.

[16] B. Choi and L. Hanzo, "Crest factors of complementary-sequence-based multicode MC-CDMA signals," IEEE Trans. Wireless Commun., vol. 2, no. 6, pp. 1114–1119, Nov. 2003.

[17] T. Jiang and G. Zhu, "Complement block coding for reduction in peak-to-average power ratio of OFDM signals," IEEE Commun. Mag., vol. 43, no. 9, pp. S17-S22, Sep. 2005.

[18] L. Yang, M. You and J. Li, "Optimized Spreading Code Reallocation Technique for PAPR Reduction in MC-CDMA systems," Proc. 2009 IEEE Global Communications Conference, 30 Nov.-4 Dec. 2009, Honolulu, HI, USA. DOI: 10.1109/GLOCOM.2009.5425411.

[19] Y. Hori and H. Ochiai, "A Low PAPR Subcarrier Hopping Multiple Access with Coded OFDM for Low Latency Wireless Networks," Proc. 2015 IEEE Global Communications Conference, 6-10 Dec. 2015, San Diego, CA, USA. DOI: 10.1109/GLOCOM.2015.7417748.

[20] A. Sahin, X. Wang, H. Lou and R. Yang, "Low-PAPR Multi-channel OOK Waveform for IEEE 802.11ba Wake-up Radio," Proc. 2019 IEEE Global Communications Conference, 9-13 Dec. 2019, Waikoloa, HI, USA.

[21] S.-Y. Sun, H.-H. Chen, and W.-X. Meng, "A Survey on complementary-coded MIMO CDMA wireless communications," IEEE Commun. Surveys Tuts., vol. 17, no. 1, pp. 52–69, 1st Quart., 2015, DOI: 10.1109/COMST.2014.2332999.

[22] M. Dávidekováa *et al*., "Applications of complete complementary codes and propositions for future research areas of these codes," the 7th International Conference on Ambient Systems, Networks and Technologies. In Proc. Computer Science 83 (2016) pp. 592 – 599.

[23] D.V. Sarwate and M.B. Pursley, "Crosscorrelation properties of pseudorandom and related sequences", Proceedings of the IEEE, vol. 68, no. 5, pp. 593–619, May 1980, DOI: 10.1109/PROC.1980.11697.

[24] Kari H. A. Karkkainen, "Meaning of maximum and mean-square cross-correlation as a performance measure for CDMA code families and their influence on system capacity", IEICE Trans. Commun. E76-B(8): pp. 848–854, 1993.

[25] R. Turyn, "Ambiguity functions of complementary sequences", IEEE Trans. on Inform. Theory, vol. IT-9, pp. 46-47, Jan. 1963.